

# On the Graded Identities for Elementary Gradings in Matrix Algebras over Infinite Fields

Diogo Diniz Pereira da Silva e Silva \*

*Unidade Acadêmica de Matemática e Estatística  
Universidade Federal de Campina Grande*

*Cx. P. 10.044, 58429-970, Campina Grande, PB, Brazil  
E-mail: diogo@dme.ufcg.edu.br*

## Abstract

We find a basis for the  $G$ -graded identities of the  $n \times n$  matrix algebra  $M_n(K)$  over an infinite field  $K$  of characteristic  $p > 0$  with an elementary grading such that the neutral component corresponds to the diagonal of  $M_n(K)$ .

## 1 Introduction

The polynomial identities of the matrix algebra  $M_n(K)$  are important in the theory of PI-algebras, for example, the T-ideal of its polynomial identities  $T(M_n(K))$  arises in Kemer's structural theory of T-ideals as one of the T-prime T-ideals. However, over infinite fields, finite bases for  $T(M_n(K))$  were determined only when  $n = 2$  and  $\text{char } K \neq 2$ . In ([21]) Razmyslov determined a basis of the identities of  $M_2(K)$  with 9 elements in the case  $\text{char } K = 0$  and in ([8]) Drensky improved this result by finding a minimal basis with two identities. A basis for the identities of  $M_2(K)$  over infinite fields of characteristic  $p > 2$  was determined by Koshlukov in ([19]). He has proved

---

\*Supported by CNPq

that the same basis found in ([8]) is a basis for the identities of  $M_2(K)$  over an infinite field  $K$  of characteristic  $> 3$ , and if  $\text{char } K = 3$  one more identity is necessary. The problem of determining a finite basis for  $T(M_n(K))$  when  $n \geq 3$  is still open.

In [18] Kemer develops a theory of T-ideals analogous to the theory of ideals in commutative polynomial algebras and the concept of  $\mathbb{Z}_2$ -graded identities was a key component in this theory showing the importance of  $\mathbb{Z}_2$ -graded identities. In [26] Di Vincenzo found a basis for the  $\mathbb{Z}_2$ -graded identities of  $M_2(K)$ , over a field of characteristic 0.

Soon afterwards, the study of  $G$ -graded identities of algebras graded by an arbitrary group  $G$  became a problem of independent interest. Vasilovsky [24, 25] extended the results of [26] and determined basis for the graded identities of  $M_n(K)$  graded by the groups  $\mathbb{Z}$  and  $\mathbb{Z}_n$ , for any  $n$ , in the case  $K$  is a field of characteristic 0. Azevedo [1, 2] proved that the results of Vasilovsky also hold if  $K$  is an infinite field of characteristic  $p > 0$ .

The complete description of all possible  $G$ -gradings of  $M_n(K)$  by a finite group  $G$ , over an algebraically closed field  $K$  of characteristic 0, is given in [5]. The main result states that any such grading is the tensor product of two types of gradings: the *fine gradings* where  $\dim(M_n(K))_g \leq 1$  for all  $g \in G$  and the *elementary gradings* which are induced by a grading in the vector space  $K^n$ . For the elementary gradings the  $G$ -graded identities of  $M_n(K)$ , where  $G$  is an arbitrary group, were studied in [4] in a very general setting. Provided that the neutral component coincides with the diagonal of  $M_n(K)$  a basis of the  $G$ -graded identities was determined when  $\text{char } K = 0$ .

In this article we combine the methods of [4] and [1, 2] to prove that the same results of [4] for the graded identities of  $M_n(K)$  with elementary gradings hold for infinite fields of characteristic  $p > 0$ .

## 2 Preliminaries

In this article  $K$  denotes an infinite field and all vector spaces and algebras will be considered over  $K$ . Let  $G$  be an arbitrary group, a  $G$ -grading of an algebra  $A$  is a vector space decomposition  $A = \bigoplus_{g \in G} A_g$  such that for any  $g, h \in G$  the inclusion  $A_g A_h \subset A_{gh}$  holds. The elements of  $A_g$  are said to be *homogeneous of degree  $g$* . We denote by  $\epsilon$  the identity element of  $G$ , the component  $A_\epsilon$  is called the *neutral component*.

We denote by  $I_k$  the set  $\{1, 2, \dots, k\} \subset \mathbb{N}$ , given  $i, j \in I_n$  we denote by

$E_{ij} \in M_n(K)$  the matrix unit in which the only non-zero entry is 1 in the  $i$ -th row and  $j$ -th column. Given an  $n$ -tuple  $(g_1, \dots, g_n) \in G^n$  a  $G$ -grading in  $M_n(K)$  is determined by imposing that  $E_{ij}$  is homogeneous of degree  $g_i^{-1}g_j$ . These gradings are called *elementary gradings*.

**Proposition 2.1 ([15])** *Let  $G$  be a group, the  $G$ -grading of  $M_n(K)$  is elementary if and only if all matrix units  $E_{ij}$  are homogeneous.*

Let  $\{X_g | g \in G\}$  be a family of disjoint countable sets indexed by  $G$  and let  $X = \cup_{g \in G} X_g$ . We denote by  $K\langle X \rangle$  the free associative algebra freely generated by  $X$ . Given a monomial  $m = x_{i_1} \dots x_{i_k}$  we denote by  $h(m)$  the  $k$ -tuple  $(h_1, \dots, h_k)$  where for each  $l \in I_k$  the  $l$ -th coordinate is the element  $h_l \in G$  such that  $x_{i_l} \in X_{h_l}$ . We denote by  $K\langle X \rangle_g$  the subspace of  $K\langle X \rangle$  generated by the monomials  $x_{j_1} \dots x_{j_k}$  such that  $h_1 \cdot \dots \cdot h_k = g$ , where  $(h_1, \dots, h_k) = h(m)$ . The decomposition  $K\langle X \rangle = \bigoplus_{g \in G} K\langle X \rangle_g$  is a  $G$ -grading and with this grading  $K\langle X \rangle$  is the free  $G$ -graded associative algebra on  $X$ . In this grading any monomial  $m = x_{i_1} \dots x_{i_k}$  is homogeneous and its degree with respect to the  $G$ -grading will be denoted by  $\alpha(m)$ .

A polynomial  $f(x_1, \dots, x_k) \in K\langle X \rangle$  is a *graded polynomial identity* for the  $G$ -graded algebra  $A$  if  $f(a_1, \dots, a_k) = 0$  whenever  $a_l \in A_{h(x_l)}$  for every  $l \in I_k$ . We denote by  $T_G(A)$  the set of all graded identities of the  $G$ -graded algebra  $A$ , this set is an ideal of  $K\langle X \rangle$  which is invariant under all graded endomorphisms of  $K\langle X \rangle$ . It is easy to show that the intersection of a family of  $T_G$ -ideals of  $K\langle X \rangle$  is also a  $T_G$ -ideal, hence given  $S \subset K\langle X \rangle$  we may define the  $T_G$ -ideal generated by  $S$ , denoted by  $\langle S \rangle^{T_G}$ , as the intersection of all  $T_G$ -ideals that contain  $S$ . We say that  $S$  is a basis of the graded identities of  $A$  if  $T_G(A) = \langle S \rangle^{T_G}$ .

In this article, with the exception of Proposition 4.1 and Remark 4.2, we fix an  $n$ -tuple  $\mathbf{g} = (g_1, \dots, g_n) \in G^n$  of pairwise different elements and let  $M_n(K) = \bigoplus_{g \in G} (M_n(K))_g$  be the elementary grading induced by  $\mathbf{g}$ .

### 3 Generic Matrices

In this section we define generic matrices and construct a relatively free algebra in the class determined by  $M_n(K)$  with an elementary grading induced by  $\mathbf{g} = (g_1, \dots, g_n) \in G^n$ . We also prove some results that will be used in the next sections.

For each  $h \in G$  let  $Y_h = \{y_{h,i}^k, 1 \leq k \leq n, i = 1, 2, \dots\}$  be a countable set of commuting variables. Let  $Y = \cup_{h \in G} Y_h$  and denote by  $\Omega = K[Y]$  the commutative polynomial algebra generated by  $Y$ . The set  $\{g_1, \dots, g_n\}$  is denoted by  $G_n$ .

The algebra  $M_n(\Omega)$  has an elementary grading induced by the  $n$ -tuple  $\mathbf{g} = (g_1, \dots, g_n)$ , given  $h \in G$  we are interested in determining the elementary matrices  $E_{ij}$  of degree  $h$ . If we fix  $i \in I_n$  there exists an elementary matrix  $E_{ij}$  of degree  $h$  if and only if  $g_i h \in G_n$  and in this case  $j \in I_n$  is determined by the equality  $g_j = g_i h$ .

For each  $h \in G$  we denote by  $L_h$  the set of all indexes  $k \in I_n$  such that  $g_k h \in G_n$  and by  $s_h^k \in I_n$  is the index determined by  $g_k h = g_{s_h^k}$ . Then it is easy to see that  $(M_n(\Omega))_h = 0$  iff  $L_h = \emptyset$ , moreover if  $L_h \neq \emptyset$  then  $\{E_{k s_h^k} | k \in L_h\}$  is the set of all elementary matrices of degree  $h$ . We consider in  $M_n(\Omega)$  the homogeneous matrices

$$A_i^{(h)} = \sum_{k \in L_h} y_{h,i}^k E_{k, s_h^k}. \quad (1)$$

These matrices will be called *generic matrices*, the subalgebra  $F$  of  $M_n(\Omega)$  generated by the generic matrices is a graded subalgebra, in this grading the generic matrix  $A_i^h$  defined above is homogeneous of degree  $h$ .

**Lemma 3.1** *The relatively free algebra  $K\langle X \rangle / T_G(M_n(K))$  is isomorphic to the algebra  $F$ .*

*Proof.* The proof is analogous to that of [2, Lemma 1]. □

**Remark 3.2** *As a direct consequence we have  $T_G(F) = T_G(M_n(K))$ , so from now on we will focus on the graded identities for the algebra  $F$ .*

In order to proceed we need convenient notation to compute a product of generic matrices.

**Definition 3.3** *Let  $\mathbf{h} = (h_1, \dots, h_q) \in G^q$ , the set*

$$L_{\mathbf{h}} = \{k \in I_n \mid g_k h_1 \dots h_i \in G_n, \text{ for every } i \in I_q\}$$

*is the set associated with  $\mathbf{h}$ . For each  $k \in L_{\mathbf{h}}$  define the  $(q+1)$ -tuple  $s_k = (s_1^k, \dots, s_q^k, s_{q+1}^k)$ , inductively by setting:*

$$(1) \ s_1^k = k,$$

$$(2) \text{ for } i \in I_q \text{ the index } s_{k+1} \in I_n \text{ is determined by } g_{s_{i+1}^k} = g_{s_i^k} h_i.$$

**Remark 3.4** In the above definition  $k \in L_{\mathbf{h}}$  if and only if there exist elementary matrices  $E_{i_1 j_1}, \dots, E_{i_q j_q}$  such that  $i_1 = k$ ,  $E_{i_a j_a}$  has degree  $h_a$  and  $E_{i_1 j_1} \cdot \dots \cdot E_{i_q j_q} \neq 0$ . Moreover  $i_a = s_a^k$  and  $j_a = s_{a+1}^k$ .

**Lemma 3.5** If  $L$  is the set of indexes associated with the  $q$ -tuple  $(h_1, \dots, h_q)$  in  $G^q$  and  $s_k = (s_1^k, \dots, s_q^k, s_{q+1}^k)$  denotes the corresponding sequence determined by  $k \in L$  then

$$A_{i_1}^{h_1} \dots A_{i_q}^{h_q} = \sum_{k \in L} w_k E_{s_1^k, s_{q+1}^k}$$

$$\text{where } w_k = y_{h_1, i_1}^{s_1^k} y_{h_2, i_2}^{s_2^k} \dots y_{h_q, i_q}^{s_q^k}.$$

*Proof.* From the previous remark we conclude that

$$E_{k_1 s_{h_1}^{k_1}} E_{k_2 s_{h_2}^{k_2}} \dots E_{k_q s_{h_q}^{k_q}} \neq 0,$$

iff  $k_1 \in L$  and for every  $i \in I_q$  we have  $k_i = s_i^k$ ,  $s_{h_i}^{k_i} = s_{i+1}^k$ . Since from (1) we have

$$A_{i_1}^{h_1} \dots A_{i_q}^{h_q} = \sum (y_{h_1, i_1}^{k_1} \dots y_{h_q, i_q}^{k_q}) E_{k_1, s_{h_1}^{k_1}} \dots E_{k_q, s_{h_q}^{k_q}},$$

the result follows.  $\square$

**Remark 3.6** In order to simplify the notation we will adopt the following convention. If  $f(x_{i_1}, \dots, x_{i_k}) \in K\langle X \rangle$  is a polynomial in the variables  $x_{i_1}, \dots, x_{i_k}$  then  $A_{i_l}$  will denote the generic matrix  $A_{i_l}^{h(x_{i_l})}$ , and  $f(A_{i_1} \dots A_{i_k})$  denotes the result of substituting each variable for the corresponding generic matrix.

The following two consequences of the above lemma will be useful in the next section. We recall that given a monomial  $m = x_{i_1} \dots x_{i_q}$  of length  $q$ , the sequence  $(h_1, \dots, h_q)$  where  $h_k$  is the degree of the variable  $x_{i_k}$ , is denoted by  $h(m)$  and  $\alpha(m)$  denotes the degree of the monomial in the  $G$ -grading of  $K\langle X \rangle$  defined in Section 2.

**Corollary 3.7** *Let  $m_1, m_2$  be monomials such that  $h(m_1) = h(m_2)$ , then  $m_1 \in T_G(F)$  if and only if  $m_2 \in T_G(F)$ .*

*Proof.* It follows directly from Lemma 3.5 above since  $m_i \in T_G(F)$  if and only if the set associated with  $h(m_i)$  is empty.  $\square$

**Corollary 3.8** *If  $x_{i_1}x_{i_2}\dots x_{i_r}$  and  $x_{j_1}x_{j_2}\dots x_{j_s}$  are two monomials such that the matrices  $A_{i_1}\dots A_{i_r}$  and  $A_{j_1}\dots A_{j_s}$  have in the same position the same non-zero entry then  $r = s$  and there exists  $\sigma \in S_r$  such that*

$$x_{j_1}x_{j_2}\dots x_{j_s} = x_{i_{\sigma(1)}}x_{i_{\sigma(2)}}\dots x_{i_{\sigma(r)}},$$

and moreover for every  $l \in I_r$

$$\alpha(x_{i_{\sigma(1)}}\dots x_{i_{\sigma(l-1)}}) = \alpha(x_{i_1}\dots x_{i_{l-1}}).$$

*Proof.* Let us assume the matrices  $A_{i_1}\dots A_{i_r}$  and  $A_{j_1}\dots A_{j_s}$  have the same non-zero entry in the position  $(k, l)$ , let  $\mathbf{h}(m) = (h_1^m, \dots, h_r^m)$  and  $\mathbf{h}(n) = (h_1^n, \dots, h_s^n)$ . It follows from Lemma 3.5 that  $k \in L_{\mathbf{h}(m)} \cap L_{\mathbf{h}(n)}$ , moreover the two monomials in  $\Omega$

$$y_{h_1^m, i_1}^{s_1^{k,m}} \cdot y_{h_2^m, i_2}^{s_2^{k,m}} \cdot \dots \cdot y_{h_r^m, i_r}^{s_r^{k,m}}$$

and

$$y_{h_1^n, j_1}^{s_1^{k,n}} \cdot y_{h_2^n, j_2}^{s_2^{k,n}} \cdot \dots \cdot y_{h_s^n, j_s}^{s_s^{k,n}}$$

are equal, where  $s^{k,m} = (s_1^{k,m}, s_2^{k,m}, \dots, s_r^{k,m})$  (resp.  $s^{k,n}$ ) denotes the sequence corresponding to  $k \in L_{\mathbf{h}(m)}$  (resp.  $k \in L_{\mathbf{h}(n)}$ ).

From the equality of the monomials we conclude that  $r = s$  and there exists  $\sigma \in S_r$  such that  $j_l = i_{\sigma(l)}$  for all  $l \in I_r$ , then

$$x_{j_1}x_{j_2}\dots x_{j_r} = x_{i_{\sigma(1)}}x_{i_{\sigma(2)}}\dots x_{i_{\sigma(r)}}.$$

Moreover  $s_l^{k,n} = s_{\sigma(l)}^{k,m}$ , and it follows from Definition 3.3 that

$$g_k h_{i_{\sigma(1)}} \dots h_{i_{\sigma(l-1)}} = g_k h_{i_1} \dots h_{i_{l-1}},$$

therefore  $\alpha(x_{i_{\sigma(1)}}\dots x_{i_{\sigma(l-1)}}) = \alpha(x_{i_1}\dots x_{i_{l-1}})$ .  $\square$

## 4 Graded Identities and Preliminary Results

We consider the following polynomials:

$$x_1x_2 - x_2x_1, \quad h(x_1) = h(x_2) = \epsilon, \quad (2)$$

$$x_1x_3x_2 - x_2x_3x_1, \quad \epsilon \neq h(x_1) = h(x_2) = h(x_3)^{-1}, \quad (3)$$

$$x_1 = 0, \quad (M_n(K))_{h(x_1)} = 0. \quad (4)$$

**Proposition 4.1** *Let  $\mathbf{g} = (g_1, \dots, g_n) \in G^n$  be an arbitrary  $n$ -tuple and let  $M_n(K) = \bigoplus_{g \in G} (M_n(K))_g$  be the elementary grading induced by  $\mathbf{g}$ . The following statements are equivalent:*

- (i) *If  $i \neq j$  then  $g_i \neq g_j$ , i. e., the elements in  $\mathbf{g}$  are pairwise different;*
- (ii) *The subspace  $(M_n(K))_\epsilon$  coincides with the subspace of the diagonal matrices;*
- (iii) *The polynomial  $x_1x_2 - x_2x_1$ , where  $h(x_1) = h(x_2) = \epsilon$ , is a graded identity for  $M_n(K)$ .*

*Proof.* Clearly (i) and (ii) are equivalent and (ii) implies (iii). To conclude we will prove that (iii) implies (i). Let  $E_{ij}$  be an elementary matrix of degree  $\epsilon$ , it follows from (iii) that  $E_{ij} = E_{ii}E_{ij} = E_{ij}E_{ii}$ , hence  $i = j$ .  $\square$

**Remark 4.2** *If there are  $m$  equal elements in the  $n$ -tuple it is easy to see that there is a subalgebra of  $(M_n(K))_\epsilon$  isomorphic to  $M_m(K)$  and any ordinary identity of this algebra would produce a graded identity of  $M_n(K)$  in the variables  $X_\epsilon$ .*

We recall that, with the exception of Proposition 4.1 and Remark 4.2 above, we fixed an  $n$ -tuple  $\mathbf{g} = (g_1, \dots, g_n) \in G^n$  of pairwise different elements and  $M_n(K) = \bigoplus_{g \in G} (M_n(K))_g$  is the elementary grading induced by  $\mathbf{g}$ .

**Lemma 4.3** *The  $G$ -graded algebra  $M_n(K)$  satisfies the  $G$ -graded polynomial identities (2), (3) and (4).*

*Proof.* [4, Lemma 4.1]. □

**Definition 4.4** We denote by  $J$  the  $T_G$ -ideal generated by the identities (2), (3) and (4).

**Lemma 4.5** Let  $\overline{m}(x_1, \dots, x_q)$  and  $\overline{n}(x_1, \dots, x_q)$  be two monomials that start with the same variable and let  $m(x_1, \dots, x_q)$ ,  $n(x_1, \dots, x_q)$  be the monomials obtained from  $\overline{m}$  and  $\overline{n}$  respectively by deleting the first variable. If there exist matrices  $A_1, \dots, A_q$ , such that  $\overline{m}(A_1, \dots, A_q)$  and  $\overline{n}(A_1, \dots, A_q)$  have in the same position the same non-zero entry then  $m(A_1, \dots, A_q)$  and  $n(A_1, \dots, A_q)$  also have in the same position the same non-zero entry.

*Proof.* It follows directly from Lemma 3.5. □

In the next lemma we follow the idea of Azevedo, see [1, Lemma 6] and [2, Lemma 5].

**Lemma 4.6** Let  $m(x_1, \dots, x_p)$  and  $n(x_1, \dots, x_p)$  be two monomials such that the matrices  $n(A_1, \dots, A_p)$  and  $m(A_1, \dots, A_p)$  have in the same position the same non-zero entry then

$$m(x_1, x_2, \dots, x_p) \equiv n(x_1, x_2, \dots, x_p) \text{ modulo } J.$$

*Proof.* Let  $m(x_1, \dots, x_p) = x_{i_1} \dots x_{i_q}$ , for integers  $1 \leq k < l \leq q + 1$  we denote by  $m^{[k, l]} = x_{i_k} \dots x_{i_{l-1}}$  (resp.  $n^{[k, l]}$ ) the monomial obtained from  $m$  (resp.  $n$ ) by deleting the first  $k - 1$  variables and the last  $q - l + 1$  variables.

It follows from Corollary 3.8 that there exists  $\sigma \in S_q$  such that

$$n(x_1, \dots, x_p) = x_{i_{\sigma(1)}} \dots x_{i_{\sigma(q)}}, \quad (5)$$

and

$$\alpha(n^{[1, l]}) = \alpha(m^{[1, \sigma(l)]}), \quad (6)$$

for all  $l \in I_q$ . We will prove the result by induction on  $q$ , if  $q = 1$  the lemma is clearly true.

Let  $a = \sigma^{-1}(1)$ , it follows from (6) that

$$\alpha(n^{[1, a]}) = \epsilon.$$

Assume there exists  $r \in I_{q-1}$  such that

$$\sigma^{-1}(r) < a < \sigma^{-1}(r + 1).$$



The result follows from the previous lemma and the induction hypothesis if we prove that in this case  $n(x_1, \dots, x_p)$  is congruent modulo  $J$  to a monomial that starts with  $x_{i_1}$ . It follows from (6) that

$$\alpha(n^{[\sigma^{-1}(r), \sigma^{-1}(r+1)]}) = \alpha(m^{[r, r+1]}) = \alpha(x_{i_r}).$$

Using (5) we obtain  $n^{[\sigma^{-1}(r), \sigma^{-1}(r+1)]} = x_{i_r} n^{[\sigma^{-1}(r)+1, \sigma^{-1}(r+1)]}$ , therefore

$$\alpha(n^{[\sigma^{-1}(r)+1, \sigma^{-1}(r+1)]}) = \epsilon.$$

Finally if  $\sigma^{-1}(r) + 1 = a$  then using the identity (2) we have

$$n = n^{[1, a]} n^{[a, \sigma^{-1}(r+1)]} \equiv_J n^{[a, \sigma^{-1}(r+1)]} n^{[1, a]} = x_{i_1} n^{[a+1, \sigma^{-1}(r+1)]} n^{[1, a]}.$$

The other possibility is  $\sigma^{-1}(r) + 1 < a$ , in this case we have

$$n = n^{[1, \sigma^{-1}(r)+1]} n^{[\sigma^{-1}(r)+1, a]} n^{[a, \sigma^{-1}(r+1)]},$$

since

$$\alpha(n^{[1, \sigma^{-1}(r)+1]} n^{[\sigma^{-1}(r)+1, a]}) = \alpha(n^{[\sigma^{-1}(r)+1, a]} n^{[a, \sigma^{-1}(r+1)]}) = \epsilon$$

it follows from (3) that  $n \equiv_J n^{[a, \sigma^{-1}(r+1)]} n^{[\sigma^{-1}(r)+1, a]} n^{[1, \sigma^{-1}(r)+1]}$ , and the first variable in this last monomial is  $x_{i_1}$ . Hence this lemma is proved under the assumption that exists  $r \in I_{q-1}$  such that  $\sigma^{-1}(r) < a < \sigma^{-1}(r+1)$ .

If there exists no such  $r$  it follows that  $\sigma(I_{a-1}) = \{b, b+1, \dots, q\}$  for some  $b \in I_q$ , and in this case the monomials  $m^{[b, q+1]}$  and  $n^{[1, a]}$  have the same multidegree and in particular  $\alpha(m^{[b, q+1]}) = \alpha(n^{[1, a]}) = \epsilon$ . Moreover  $\sigma(1) \geq b$  and it follows from (6) that  $\alpha(m^{[1, \sigma(1)]}) = \alpha(n^{[1, 1]}) = \epsilon$ , hence

$$\alpha(m^{[1, b]} m^{[b, \sigma(1)]}) = \alpha(m^{[b, \sigma(1)]} m^{[\sigma(1), q+1]}) = \epsilon$$

and since  $m = m^{[1, b]} m^{[b, \sigma(1)]} m^{[\sigma(1), q+1]}$ , it follows from (2) that

$$m \equiv_J m^{[\sigma(1), q+1]} m^{[b, \sigma(1)]} m^{[1, b]},$$

and this last monomial starts with the same variable as  $n$ . □

## 5 A basis for the graded identities of $M_n(K)$

In [1, 2] it is shown that there are no nontrivial identities  $x_{i_1} \dots x_{i_k}$  for  $M_n(K)$  with its natural elementary  $\mathbb{Z}_n$  and  $\mathbb{Z}$  gradings respectively, however in [4, Example 4.7, Theorem 4.8, Theorem 4.9] concrete gradings are considered where  $M_n(K)$  satisfies nontrivial identities  $x_{i_1} \dots x_{i_k}$ . These results also hold for infinite fields of positive characteristic.

In this section in Theorem 5.2 we prove that the main result of [4] holds for arbitrary infinite fields. This theorem states that for the elementary gradings considered here a basis for the graded identities of  $M_n(K)$  consists of (2)-(4), and a finite number of monomials  $x_{i_1} \dots x_{i_k}$  of length  $k$  bounded by a function of  $n$ . The following lemma corresponds to [4, Proposition 4.2].

**Lemma 5.1** *Let  $G_0 = \{g \in G \mid (M_n(K))_g \neq 0\}$  be the support of a  $G$ -grading of  $M_n(K)$  and let  $I$  be the set of all finite sequences  $(h_1, \dots, h_k)$  of elements of  $G_0$  such that the monomial  $m$  with  $h(m) = (h_1, \dots, h_k)$  is a  $G$ -graded polynomial identity of  $M_n(K)$ . Then there exists a positive integer  $n_0$  such that  $m$  is a consequence of the  $G$ -graded polynomial identities of  $M_n(K)$  as in (2), (3) and (4) together with the monomials  $m$  of length  $k$ , where  $h(m) \in I$  and  $k < n_0$ .*

*Proof* Let  $s = |G_0|$ ,  $n_0 = 4s^{2s+2}$  and let  $U$  be the  $T_G$ -ideal generated by (2)-(4) and the monomials  $m$  of length less than  $n_0$  such that  $h(m) \in I$ . If  $m \in T_G(F)$  is a multilinear monomial then it follows from [4, Proposition 4.2] that  $m \in U$ . If  $m$  is not multilinear let  $\bar{m}$  be a multilinear monomial such that  $h(m) = h(\bar{m})$ . It follows from Corollary 3.7 that  $\bar{m} \in T_G(F)$ , then  $\bar{m}$  is in  $U$  and since  $m$  is obtained from  $\bar{m}$  by identifying some of the variables we conclude that  $m \in U$ .  $\square$

**Theorem 5.2** *Let  $K$  be an infinite field. Let  $G$  be any group and let  $g = (g_1, \dots, g_n) \in G^n$  induce an elementary  $G$ -grading of  $M_n(K)$  where the elements  $g_1, \dots, g_n$  are pairwise different. Then a basis of the graded polynomial identities of  $M_n(K)$  consists of (2)-(4) and a finite number of identities of the form  $x_{i_1} \dots x_{i_k}$ ,  $k \geq 2$ , where the length  $k$  is bounded by a function of  $n$ .*

*Proof.* Let  $U$  be the  $T_G$ -ideal defined in the previous lemma, since  $|G_0| = s \leq n^2$  it is easy to see that we may choose  $k \leq 4n^{4(n^2+1)}$ . It follows from Lemma 4.3 that  $U \subset T_G(F)$ . We wish to prove that  $T_G(F) \subset U$ , suppose on

contrary that the inclusion does not hold. Since the field  $K$  is infinite, there exists a multihomogeneous identity of  $F$  that does not belong to  $U$ . Let

$$f(x_1, \dots, x_n) = \sum_{i=1}^{k_0} a_i m_i(x_1, \dots, x_n),$$

where  $a_i \neq 0$ ,  $1 \leq i \leq k_0$ , be a multihomogeneous element of  $T_G(F) - U$  with the minimal number of non-zero summands. If  $k_0 = 1$  then the monomial  $m_1$  is an identity of  $F$  that is not an element of the  $T_G$ -ideal  $U$ , but this contradicts Lemma 5.1, hence  $k_0 > 1$ . Moreover, since  $k_0$  is minimal we conclude that  $m_1$  is not an identity. Clearly,

$$-a_1 m_1(A_1, \dots, A_n) = \sum_{i=2}^{k_0} a_i m_i(A_1, \dots, A_n).$$

The non-zero entries in each matrix  $m_i(A_1, \dots, A_n)$  are monomials in  $\Omega$  and  $m_1(A_1, \dots, A_n) \neq 0$ , hence there exists  $p \in \{2, \dots, k_0\}$  such that the matrices  $m_1(A_1, \dots, A_n)$  and  $m_p(A_1, \dots, A_n)$  have in the same position the same nonzero entry. Using Lemma 4.6 we conclude that  $m_p - m_1 \in U$ . In this case

$$f(x_1, \dots, x_n) + a_p(m_1(x_1, \dots, x_n) - m_p(x_1, \dots, x_n)) \in T_G(F) - U,$$

but the last polynomial is a sum of  $k_0 - 1$  monomials and this is a contradiction since  $k_0$  is minimal.  $\square$

The following corollary generalizes the main result of [1] for any elementary grading of  $M_n(K)$  in which the neutral component consists of the diagonal matrices and  $|G| = n$ .

**Corollary 5.3** *Let  $K$  be an infinite field. Let  $G$  be a finite group of order  $n$  and let  $g = (g_1, \dots, g_n) \in G^n$  induce an elementary  $G$ -grading of  $M_n(K)$  where the elements  $g_1, \dots, g_n$  are pairwise different. Then a basis of the graded polynomial identities of  $M_n(K)$  consists of (2) and (3).*

*Proof.* Since  $|G| = n$  the set  $G_n$  in Definition 3.3 is the group  $G$ , therefore for every  $q \in \mathbb{N}$  and every sequence  $\mathbf{h} \in G^q$  the set  $L_{\mathbf{h}}$  associated with this sequence is  $I_n = \{1, 2, \dots, n\}$ . Hence Lemma 3.5 implies no monomial  $x_{i_1} \dots x_{i_k} \in K\langle X \rangle$  of length  $k \geq 1$  is a graded identity for  $M_n(K)$  and the corollary now follows from the previous theorem.  $\square$

## Acknowledgements

We thank the referee for valuable remarks. Thanks are due to P. Koshlukov for his useful suggestions.

## References

- [1] S. S. Azevedo, *Graded identities for the matrix algebra of order  $n$  over an infinite field*, Comm. Algebra **30**, 5849–5860, 12 (2002).
- [2] S. S. Azevedo, *A basis for  $\mathbb{Z}$ -graded identities of matrices over infinite fields*, Serdica Math. Journal **29** (2), 149–158 (2003).
- [3] S. S. Azevedo, P. Koshlukov, *Graded identities for  $T$ -prime algebras over field of positive characteristic*, Israel J. Math. **128**, 157–176 (2002).
- [4] Y. Bahturin, V. Drensky, *Graded polynomial identities of matrices*, Linear Algebra and its Applications, 15–34 (2002).
- [5] Yu. A. Bakhturin, M. V. Zaicev, *Group Gradings on Matrix Algebras*, Candad. Math. Bull. **45** (4), 499–508 (2002).
- [6] A. Berele, *Magnum PI*, Israel J. Math. **51**, no. 1-2, 13–19, (1985).
- [7] A. Ya. Belov, L. H. Rowen, *Computational Aspects of polynomial identities*, Research Notes in Mathematics **9**, A.K. Peters, Ltd., Wellesley, MA, 2005.
- [8] V. Drensky, *A minimal basis of identities for a second-order matrix algebra over a field of characteristic 0*, Algebra and Logic **20**(3), 188–194(1981).
- [9] V. Drensky, *Free algebras and PI algebras*, Graduate Course in Algebra, Springer-Verlag PTE.LTD, (1999).
- [10] V. Drensky, E. Formanek, *Polynomial Identity Rings*, CRM Advanced Courses in Mathematics, Birkhäuser, Basel (2004).
- [11] E. Formanek, *The ring of generic matrices*, J. Algebra, **258**, no. 1, 310–320 (2002).
- [12] G. K. Genov, *A basis for identities of third order matrix algebra over a finite field*, Algebra and Logic, **20**, 241–257(1981).
- [13] G. K. Genov, P.N. Siderov, *A basis for identities of the algebra of fourth-order matrices over a finite field I, II* Serdica 8, 313–323, 351–366(1982).

- [14] A. Giambruno, M. Zaicev, *Polynomial identities and asymptotic methods*, Math. Surv. and Monographs **122**, AMS.
- [15] S. Dăscălescu, B. Ion, C. Năstăsescu, J. Rios Montes, *Group gradings on full matrix rings*, J. Algebra **220**, 709–728, 1999.
- [16] N. Jacobson, *PI-Algebras, an introduction*, Springer Lecture Notes in Math. **441**, Springer, (1975).
- [17] I. Kaplansky, *Rings with a polynomial identity*, Bull. Amer. Math. Soc. **54** **220**, 496–500, 1948.
- [18] A. Kemer, *Finite basis property of identities of associative algebras*, Algebra and Logic **26**, 362–397(1987).
- [19] P. Koshlukov, *Basis of the identities of the matrix algebra of order two over a field of characteristic  $p \neq 2$* , J. Algebra **241**, 410–434 (2001).
- [20] Yu. N. Maltsev, E. N. Kuzmin, *A basis for the identities of the algebra of second-order over a finite field*, Algebra and Logic **17** No. 1, 18–21(1978).
- [21] Yu. P. Razmyslov, *Finite basing of the identities of a matrix algebra of second order over a field of characteristic zero*, Algebra and Logic **12**, 47–63(1973).
- [22] Yu. P. Razmyslov, *Trace Identities of full matrix algebras over a field of characteristic zero*, Izv. Akad. Nauk SSSR, Ser. Mat. **38**, 723–756 (1974).
- [23] L. H. Rowen, *Polynomial Identities in Ring Theory*, Acad. Press Pure and Applied Math., vol. 84, New York (1980).
- [24] S. Yu. Vasilovsky,  *$\mathbb{Z}$ -graded polynomial identities of the full matrix algebra*, Commun. Algebra **26** (2), 601–612 (1998).
- [25] S. Yu. Vasilovsky,  *$\mathbb{Z}_n$ -graded polynomial identities of the full matrix algebra of order  $n$* , Proc. Amer. Math. Soc. **127** (12), 3517–3524 (1999).
- [26] O. M. Di Vincenzo, *On the graded identities of  $M_{1,1}(E)$* , Isr. J. Math. **80**, 323–335 (1992).
- [27] O. M. Di Vincenzo, *On the graded identities of  $M_{1,1}(E)$* , Isr. J. Math. **80**, 323–335 (1992).